

# Duality invariance implies Poincaré invariance

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We consider all possible dynamical theories which evolve two transverse vector fields out of a three-dimensional Euclidean hyperplane, subject to only two assumptions: (i) the evolution is local in space, and (ii) the theory is invariant under “duality rotations” of the vector fields into one another. The commutators of the Hamiltonian and momentum densities are shown to be necessarily those of the Poincaré group or its zero signature contraction. Space-time structure thus emerges out of the principle of duality.

The original description of electricity and magnetism devised by Faraday was formulated in terms of electric and magnetic lines of force. In its simplest and purest form, the lines of force never end. There are no sources. In contemporary mathematical parlance, the electric and magnetic fields are divergence-free. In its most generic formulation, the principle of electric-magnetic duality states that the electric field density  $\mathcal{E}^i$  and magnetic field density  $\mathcal{B}^i$  are to be treated on the same footing. It is then natural to introduce a notation which enables one to easily keep track of their interchange. One thus writes,

$$(\mathcal{B}_a^i) = (\mathcal{B}^i, \mathcal{E}^i), \quad a = 1, 2, \quad (1)$$

and gives the following precise meaning to the expression “on the same footing”: one demands rotational invariance in the two-dimensional plane whose axes are labelled by the index  $a$ . These rotations are termed “duality rotations”.

If one demands that the  $\mathcal{B}_a^i$  should have a dynamical evolution with a Hamiltonian structure, one needs to introduce a Poisson bracket among them. The simplest possibility that is invariant under duality rotations and spatial rotations, is local in space, and is consistent with the divergence-free character,

$$\mathcal{B}_{a,i}^i = 0 \quad (2)$$

of  $\mathcal{B}_a^i$  is,

$$[\mathcal{B}_a^i(x), \mathcal{B}_b^j(x')] = -\epsilon^{ijk} \epsilon_{ab} \delta_{,k}(x, x'), \quad (3)$$

and it is actually unique [1]. It follows from (3) that the Poisson bracket of two duality-invariant quantities is duality invariant.

We will call “local” an expression where the  $\mathcal{B}_a^i$  appear undifferentiated (sometimes this very restricted notion is called “ultralocal” because no derivatives are admitted). Two duality invariant quantities that will play a key role in what follows are the scalar

$$h = \frac{1}{2} \mathcal{B}_a^i \mathcal{B}_b^j \delta^{ab} \delta_{ij} \quad (4)$$

and the vector

$$\mathcal{H}_k = -\frac{1}{2} \mathcal{B}_a^i \mathcal{B}_b^j \epsilon^{ab} \epsilon_{ijk}. \quad (5)$$

From the bracket (3), one may immediately verify that the Lie derivative of any functional  $F[\mathcal{B}_a^i]$  along a spatial vector field  $\xi^i(x)$  is given by

$$\mathcal{L}_\xi F = [F, \int d^3x \xi^i(x) \mathcal{H}_i(x)]. \quad (6)$$

This means that  $\mathcal{H}_i(x)$  is the momentum density in curvilinear coordinates. Thus, in particular, in Cartesian coordinates  $\mathcal{H}_i(x)$  is the linear momentum density while in cylindrical coordinates  $\mathcal{H}_\varphi(x)$  is the angular momentum density around the  $z$ -axis. The  $\mathcal{H}_i(x)$  obey the Poisson bracket algebra

$$[\mathcal{H}_i(x), \mathcal{H}_j(x')] = \mathcal{H}_i(x') \delta_{,j}(x, x') + \mathcal{H}_j(x) \delta_{,i}(x, x'). \quad (7)$$

In order to introduce dynamics, we need to bring in a Hamiltonian  $H$ , which will evolve the fields off a given initial three-dimensional surface. We will demand that it be of the form

$$H = \int d^3x \mathcal{H}(x), \quad (8)$$

where  $\mathcal{H}(x)$  is a local duality and rotation invariant function constructed out of the  $\mathcal{B}_a^i$ . We will also require that the complete set  $\mathcal{H}(x)$ ,  $\mathcal{H}_i(x)$  forms an algebra of which (7) is a subalgebra.

With these requirements only, we will prove further below that,

$$[\mathcal{H}(x), \mathcal{H}(x')] = -\epsilon \delta^{ij} (\mathcal{H}_i(x') + \mathcal{H}_i(x)) \delta_{,j}(x, x') \quad (9)$$

where  $\epsilon = 0$  or  $-1$ . Once this equation is established, we have proven our point. Space-time invariance emerges out of duality invariance.

Indeed, for  $\epsilon = -1$ , Eq. (9) is precisely the commutation rule for the energy densities shown in [2, 3] to be the condition for a field theory to be Poincaré invariant (see also [4]). This equation was referred to in the concluding sentence of [3] as “*what may well be considered the most fundamental equation of relativistic quantum field theory*”.

The case  $\epsilon = 0$  has been termed the “zero-signature” case [5]. It corresponds to a spacetime geometry whose invariance group is the contraction of the Poincaré group when the speed of light goes to zero [6]. This Carroll group ( “*Now, here, you see, it takes all the running you can do, to keep in the same place*”) was first encountered in a systematic study of possible extensions of the three-dimensional Euclidean group [7]. The zero signature geometry finds an interesting application in connection with the decoupling of spatial points near the generic singularity in the early universe [8, 9]. It also has been used as the starting point, corresponding to the “free case”, for a perturbation theory in quantum gravity [10].

It is quite remarkable that  $SO(2)$  duality rotations, which are a *circular* Euclidean invariance, give raise in spacetime to *hyperbolic* Lorentz invariance. Conversely, if one changes the circular  $\delta_{ab}$  by the hyperbolic  $\eta_{ab} = \text{diag}(-1, 1)$ , all the analysis in this letter could be repeated and one would arrive at  $\epsilon = +1$  in (9), corresponding to Euclidean spacetime. Thus one sees another fascinating imprint of duality in spacetime structure. Performing a Wick rotation  $\alpha \rightarrow i\alpha$  of the duality angle implies a Wick rotation of time  $x^0 \rightarrow ix^0$ .

The technical steps of the proof of (9) are to a considerable extent given in [11] where a different problem was treated. There, *both (9) and duality invariance* were imposed to obtain restrictions on the form of  $\mathcal{H}$ . The key difference with the present work is that the commutation relations (9) *do not have to be assumed independently*, but rather are *implied by duality invariance*. A fortiori, the restrictions on the possible  $\mathcal{H}$ ’s follow therefore *from duality invariance* alone. Every duality-invariant theory is relativistic.

The proof of (9) goes as follows. First, one observes that since  $\mathcal{H}(x)$  is a local duality-invariant function, it depends only on the two invariants  $h$  given by (4) and  $v$  defined by

$$v = \mathcal{H}_k \mathcal{H}^k, \quad (10)$$

Hence,  $\mathcal{H} = f(h, v)$ . Now, the brackets between  $h$  and  $v$  that follow from the basic brackets (3) read

$$[h(x), h(x')] = \delta^{ij} (\mathcal{H}_i(x') + \mathcal{H}_i(x)) \delta_{,j}(x, x') \quad (11)$$

$$[h(x), v(x')] = 2\delta^{ij} (h(x') \mathcal{H}_i(x') + h(x) \mathcal{H}_i(x)) \delta_{,j}(x, x') + 2\mathcal{H}_{,k}^k h \delta(x, x') \quad (12)$$

$$[v(x), v(x')] = 4\delta^{ij} (v(x') \mathcal{H}_i(x') + v(x) \mathcal{H}_i(x)) \delta_{,j}(x, x') \quad (13)$$

This implies that the bracket  $[\mathcal{H}(x), \mathcal{H}(x')]$  itself is given by

$$[\mathcal{H}(x), \mathcal{H}(x')] = \delta^{ij} (F(x') \mathcal{H}_i(x') + F(x) \mathcal{H}_i(x)) \delta_{,j}(x, x') \quad (14)$$

where  $F$  is equal to

$$F = (f_h)^2 + 4h f_h f_v + 4v (f_v)^2. \quad (15)$$

Here,  $f_h$  and  $f_v$  denotes the partial derivatives of  $f$  with respect to  $h$  and  $v$ , respectively.

We thus see that the mere requirements of duality invariance, rotation invariance and locality imply that the bracket  $[\mathcal{H}(x), \mathcal{H}(x')]$  necessarily has the form (9), but with an overall coefficient  $F$  which can be at this stage a function of the dynamical variables.

If we now implement the additional condition that  $\mathcal{H}(x)$  and  $\mathcal{H}_k(x)$  should form an algebra [12], we must require that  $F$  be a constant. This yields the differential equation

$$(f_h)^2 + 4h f_h f_v + 4v (f_v)^2 = k \quad (16)$$

for the unknown function  $f$ , where  $k$  is a constant.

To complete the proof of our claim, we observe that the constant  $k$  is non negative. Indeed, its value (which does not depend on  $h$  or  $v$  since it is a constant) can be evaluated at zero values of the fields. For  $u = v = 0$ , the equation (16) reduces to  $k = (f_h)^2$ , which manifestly shows that  $k$  is non negative. If  $k = 0$ , the resulting algebra is the zero-signature algebra  $\epsilon = 0$ . If  $k > 0$ , one can set  $k = 1$  by rescaling the generators and one gets the algebra (9) with  $\epsilon = -1$  [13].

Lastly, a few comments about the solutions of (16). For  $\epsilon = -1$  ( $k$  positive) they have been studied extensively in [11, 14]. By introducing the variable  $s$  through the relation

$$h = h, \quad s^2 = h^2 - v \geq ((\mathcal{E})^2 - (\mathcal{B})^2)^2 \geq 0,$$

and defining further

$$h = U + V, \quad s = U - V,$$

the equation (16) can be cast in the simple form

$$f_U f_V = k, \tag{17}$$

which is the Hamilton-Jacobi equation in light like coordinates for a massive particle in two dimensions. For  $\epsilon = -1$ , two well-known interesting solutions in closed form are  $f = h$  corresponding to the standard Maxwell theory, and  $f = \sqrt{1 + 2h + v}$  corresponding to the Born-Infeld theory. Equation (17) was studied in its own right, independently of any consideration about electric-magnetic duality in [15].

In the Carroll case, the mass vanishes and the equation reduces to

$$f_U f_V = 0, \tag{18}$$

whose general solutions are either functions of  $U$  or functions of  $V$ .

We believe that the argument presented in this letter reinforces the view that there is a profound connection between duality invariance and spacetime structure, and that the latter may even emerge from the former.

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[1] An illuminating way of seeing the uniqueness of the bracket (3) is the following. The divergence-free character of  $\mathcal{B}^i$  implies the existence of a vector-potential  $A_i$  which is de-

terminated up to the addition of a gradient (gauge transformation). One then introduces the Hamiltonian structure through a gauge-invariant Lagrangian. We call electric field  $\mathcal{E}^i$  the negative of the canonically conjugate momentum to  $A_i$ . It is then always divergence-free and Eq. (3) holds. Note that although for Maxwell theory  $\mathcal{E}^i = F^{0i}$ , this will not be the case for more general Lagrangians such as the one of Born and Infeld. In general,  $F^{0i}$  is not divergence-free and hence it does not admit a description in terms of lines of force, although  $\mathcal{E}^i$  always does. A more common and proper terminology for  $\mathcal{E}^i$  would in fact be “electric displacement field”, but we use the words “electric field” for short since there will be no risk of confusion.

- [2] P. A. M. Dirac, “The Conditions for a Quantum Field Theory to be Relativistic,” *Rev. Mod. Phys.* **34**, 592 (1962).
- [3] J. Schwinger, “Commutation Relations and Conservation Laws,” *Phys. Rev.* **130**, 406 (1963).
- [4] C. Teitelboim, “How commutators of constraints reflect the space-time structure,” *Annals Phys.* **79**, 542 (1973).
- [5] C. Teitelboim, “The Hamiltonian Structure of Space-Time”, in: “General Relativity and Gravitation”, Vol.1 (Plenum Press, 1980) ed: A. Held.
- [6] M. Henneaux, “Geometry Of Zero Signature Space-times,” *Bull. Soc. Math. Belg.* **31**, 47 (1979).
- [7] J.-M. Levy-Leblond, “Une nouvelle limite non-relativiste du groupe de Poincaré”, *Ann. I.H.P.* **A3**, 1 (1965);  
H. Bacry and J.-M. Levy-Leblond, “Possible kinematics,” *J. Math. Phys.* **9**, 1605 (1968).  
Quote in text is from Lewis Carroll, “Through the Looking Glass and What Alice Found There”, Chapter 2, MacMillan ( London, 1871).
- [8] V.A. Belinskii, I.M. Khalatnikov and E.M. Lifshitz, “Oscillatory approach to a singular point in the relativistic cosmology”, *Adv. Phys.* **19**, 525 (1970);  
V.A. Belinskii, I.M. Khalatnikov and E.M. Lifshitz, “Construction of a general cosmological solution of the Einstein equation with a time singularity”, *Sov. Phys. JETP* **35**, 838-841 (1972).
- [9] T. Damour, M. Henneaux and H. Nicolai, “Cosmological billiards,” *Class. Quant. Grav.* **20**, R145 (2003) [hep-th/0212256].
- [10] C. Teitelboim, “Quantum Mechanics of the Gravitational Field,” *Phys. Rev. D* **25**, 3159 (1982);

- M. Henneaux, M. Pilati and C. Teitelboim, “Explicit Solution For The Zero Signature (strong Coupling) Limit Of The Propagation Amplitude In Quantum Gravity,” *Phys. Lett. B* **110**, 123 (1982).
- [11] S. Deser and O. Sarioglu, “Hamiltonian electric / magnetic duality and Lorentz invariance,” *Phys. Lett. B* **423**, 369 (1998) [hep-th/9712067].
- [12] Just as (7), the bracket  $[\mathcal{H}(x), \mathcal{H}_i(x')]$  has no dynamical content. It only generates the alteration produced in  $\mathcal{H}(x)$  by the change in  $\mathcal{B}_a^i(x)$  induced by an infinitesimal deformation  $x^i \rightarrow x^i + \xi^i(x)$  of a curvilinear coordinate system on the flat three dimensional Euclidean surface (Lie derivative  $\mathcal{L}_\xi$ ). Since  $\mathcal{H}$  contains the three dimensional Euclidean metric which - when expressed in curvilinear coordinates - becomes a function of the  $x^i$ , the function  $\mathcal{H}$  will acquire an explicit coordinate dependence which will not be felt by  $\mathcal{H}_i$ . One must thus supplement the bracket  $[\mathcal{H}(x), \mathcal{H}_i(x')]$  by an inhomogeneous term to obtain the complete Lie derivative, even in cartesian coordinates. This gives, for the scalar density  $\mathcal{H}$ ,  $\mathcal{L}_\xi \mathcal{H} = (\xi^i \mathcal{H})_{,i} = [\mathcal{H}, \int d^3 x' \mathcal{H}_i(x')] + 2 \left( \xi_{i,j} \frac{\partial \mathcal{H}}{\partial g_{ij}} \right) \Big|_{g_{ij}=\delta_{ij}}$ . No such inhomogeneous term is present in (14), which is compatible with the bracket  $[\mathcal{H}(x), \mathcal{H}_i(x')]$ , for any function  $F$ .
- [13] Similar conclusions are expected to hold for  $(2n+1)$ -form potentials in  $4(n+1)$  dimensions. The analysis would proceed along the same lines, but would become more intricate because the number of invariants grows with the dimension. The appropriate framework is provided by the non-manifestly covariant, two-potential formulation of:  
J. H. Schwarz and A. Sen, “Duality symmetric actions,” *Nucl. Phys. B* **411**, 35 (1994) [arXiv:hep-th/9304154];  
C. Bunster and M. Henneaux, “The Action for Twisted Self-Duality,” *Phys. Rev. D* **83**, 125015 (2011) [arXiv:1103.3621 [hep-th]].
- [14] G. W. Gibbons and D. A. Rasheed, “Electric - magnetic duality rotations in nonlinear electrodynamics,” *Nucl. Phys. B* **454**, 185 (1995) [hep-th/9506035];  
M. Perry and J. H. Schwarz, “Interacting chiral gauge fields in six-dimensions and Born-Infeld theory,” *Nucl. Phys. B* **489**, 47 (1997) [hep-th/9611065].
- [15] R. Courant and D. Hilbert, “Methods of Mathematical Physics,” Vol. II, Interscience Publishers (New York and London 1962), p. 91.